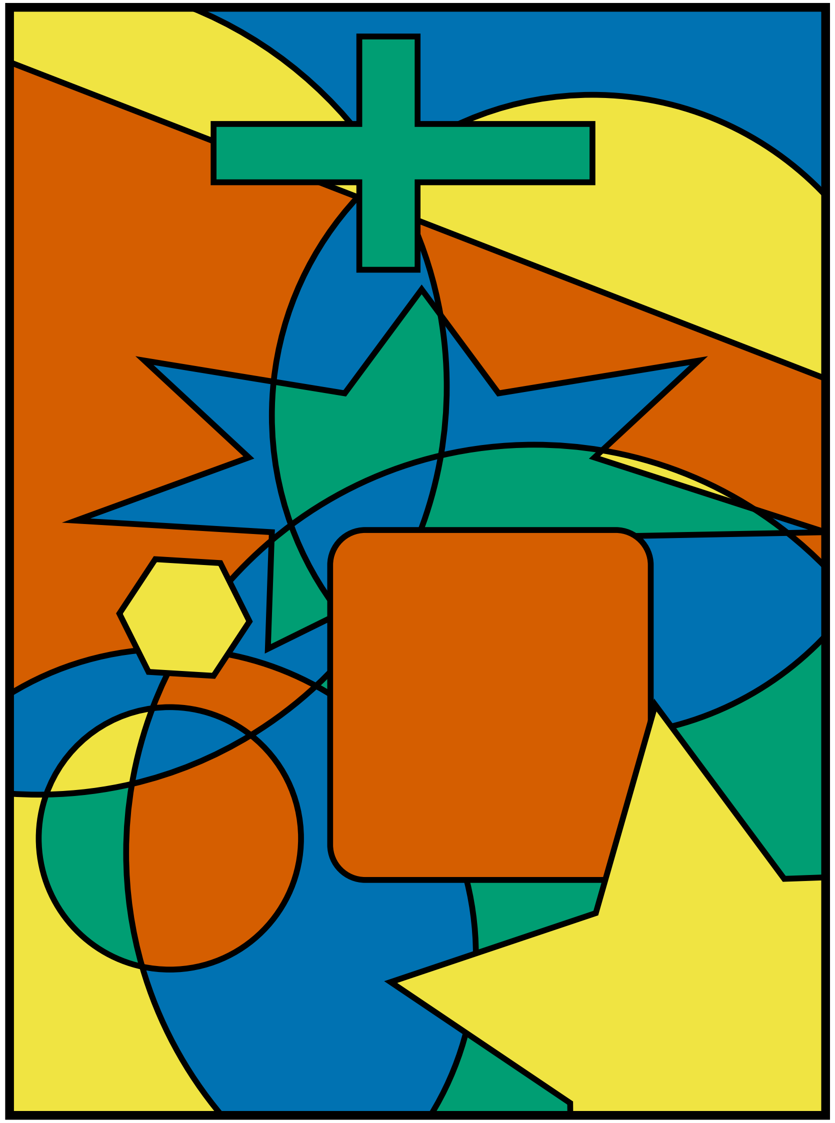
Four Colour Theorem

Any map can be coloured with a maximum of 4 colours in a way that no adjacent regions share the same colour.

Presentation by Conrad Schweiker



Task:

- **Find a 4-coloring** for figures A,B,C (use numbers 1,2,3,4)
- **Bonus:** Find a 3-coloring for figure D (use 1,2,3)

Fig. A

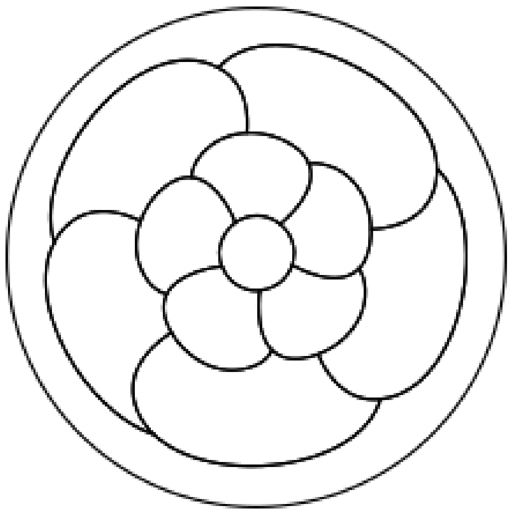


Fig. B

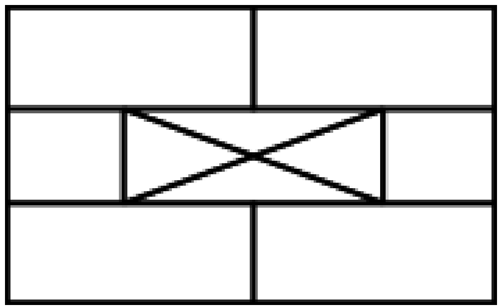


Fig. C

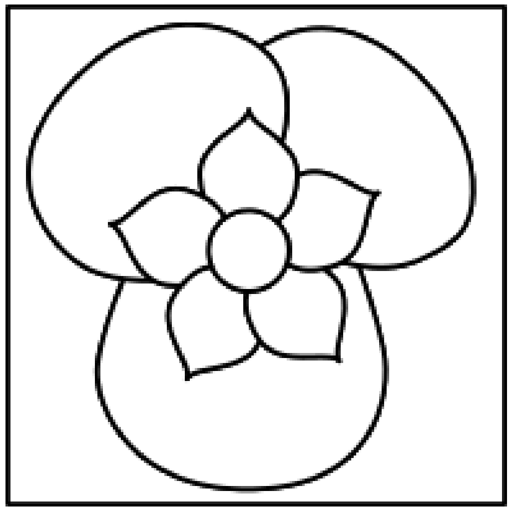
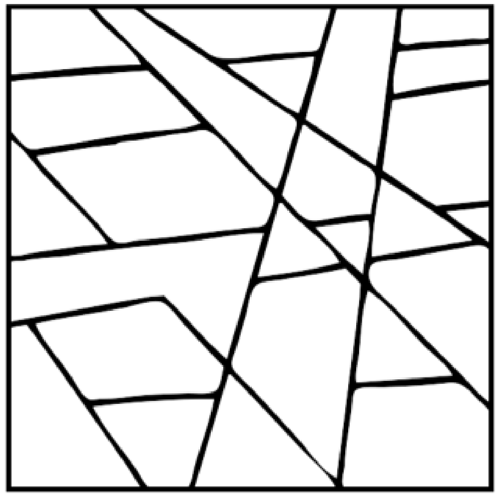


Fig. D



Example Solutions

Fig. A

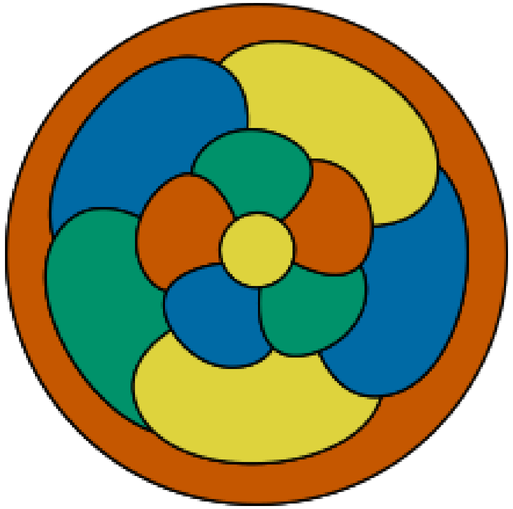


Fig. B

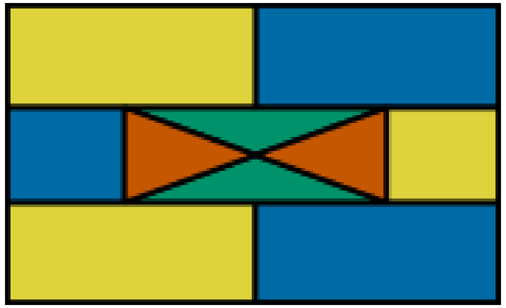


Fig. C

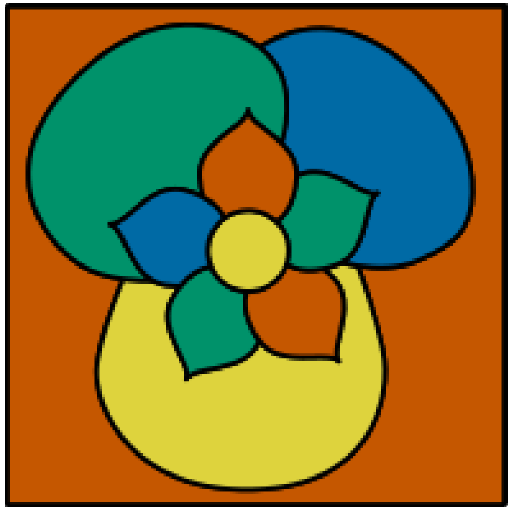
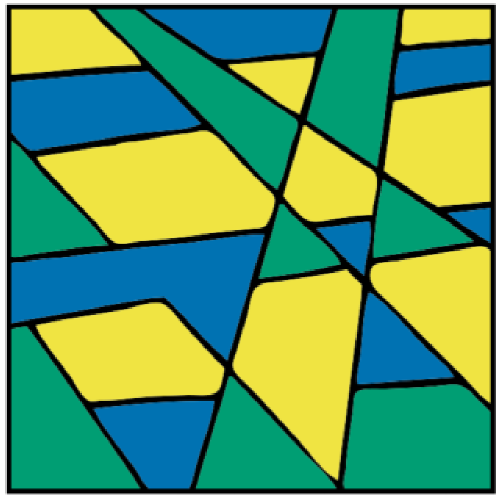


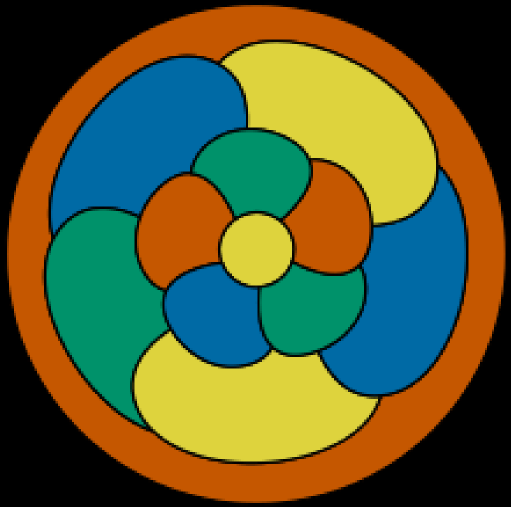
Fig. D

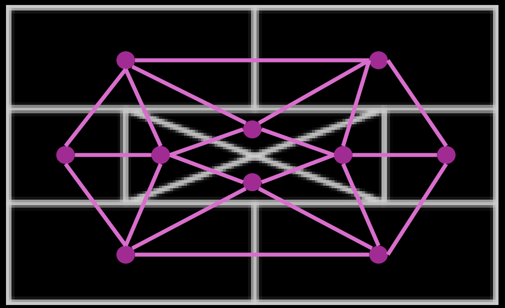


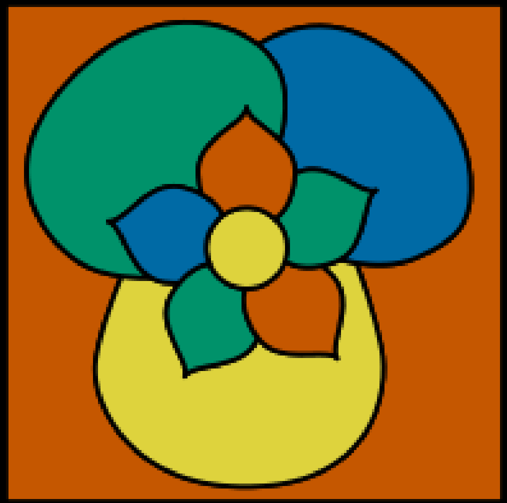
History

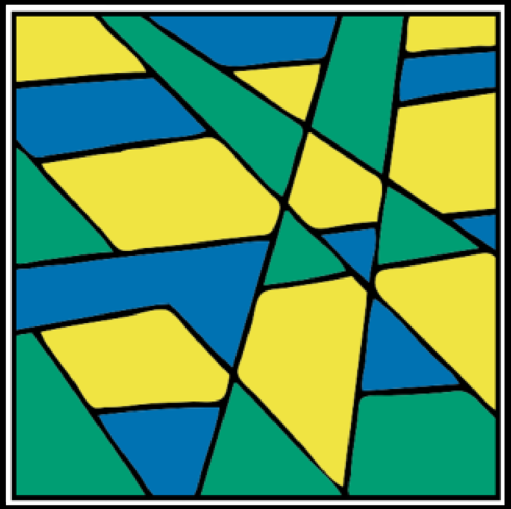
- 1852 – alleged "discovery" by Francis Guthrie
- 1879 – erroneous "proof" by Alfred Kempe
- 1890 – proof 5-colour-theorem by Heawood
- 1960-1970 – Heinrich Heesch invents solving algorithms
- 1976 – proof by Kenneth Appel & Wolfgang Haken $|U| = 1989$
- 1996 – Robertson et al. reduce to $|U| = 633$
- 2005 – Formal proof in Coq by Georges Gonthier & Benjamin Werner
- 13th of October 2024 – Human-readable proof by Carl Feghali?

- Dual Graph
- Connected Graph
- Planar Graph
- Simple Graph









Task:

- Draw the **dual graph of** figure **A and B** *next to* the figure.

Fig. A

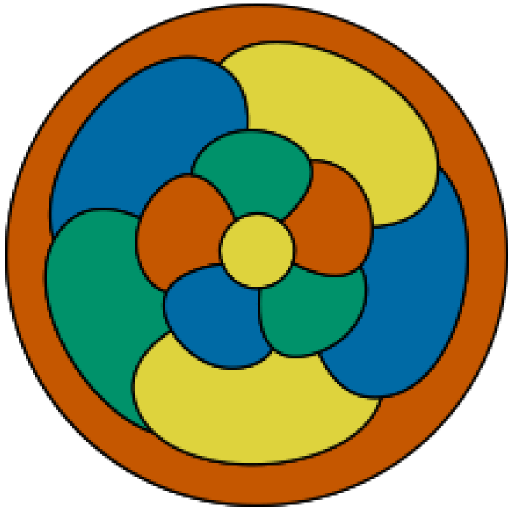


Fig. B

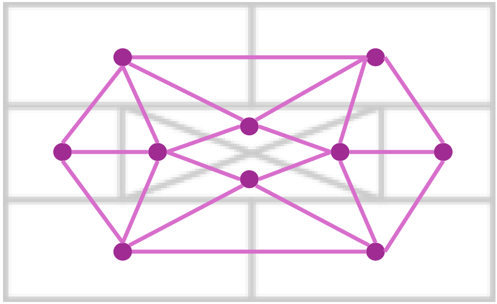


Fig. C

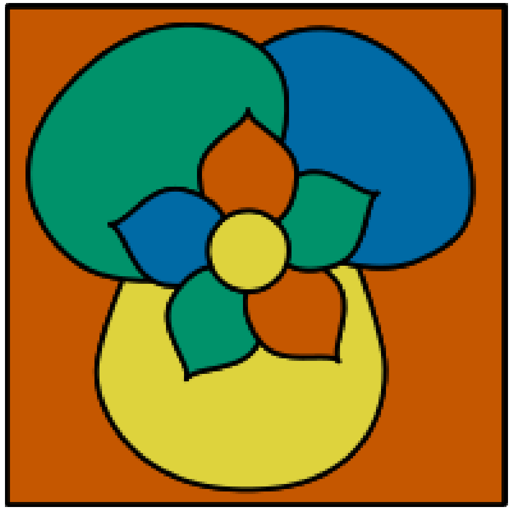
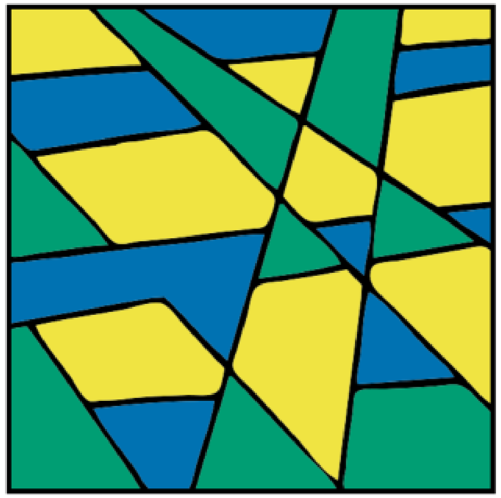


Fig. D



Solutions

Four Colour Theorem:

Any planar, connected, simple graph can be vertex-coloured in a way that two edge-connected vertices have different colours.

Fig. A

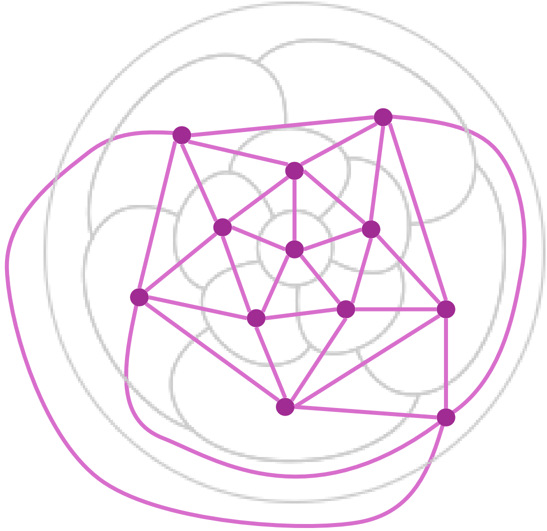


Fig. B

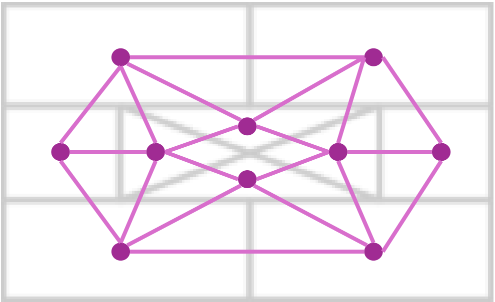


Fig. C

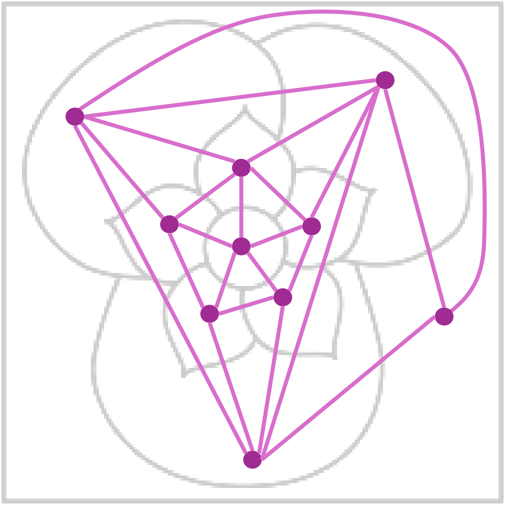
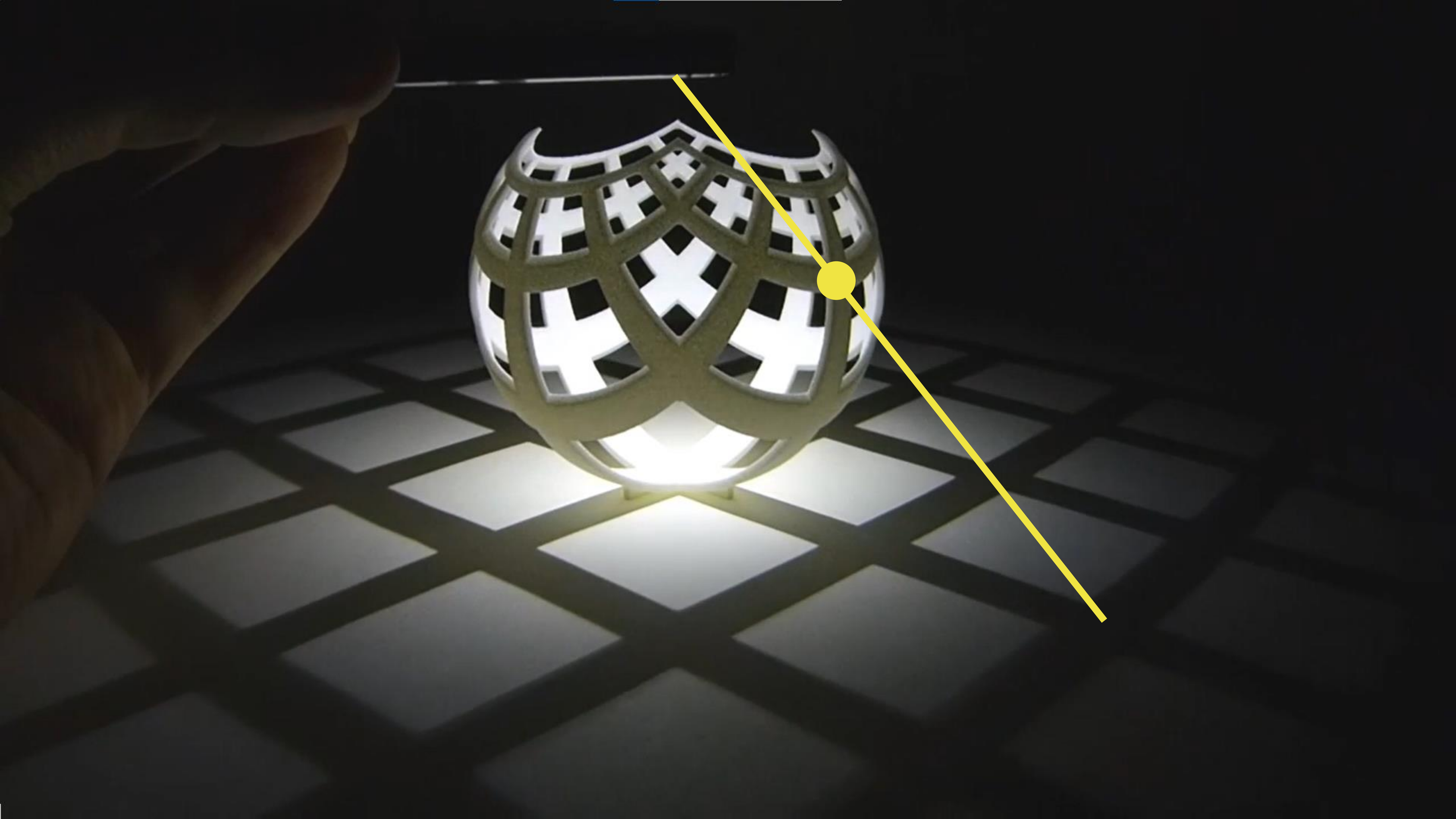
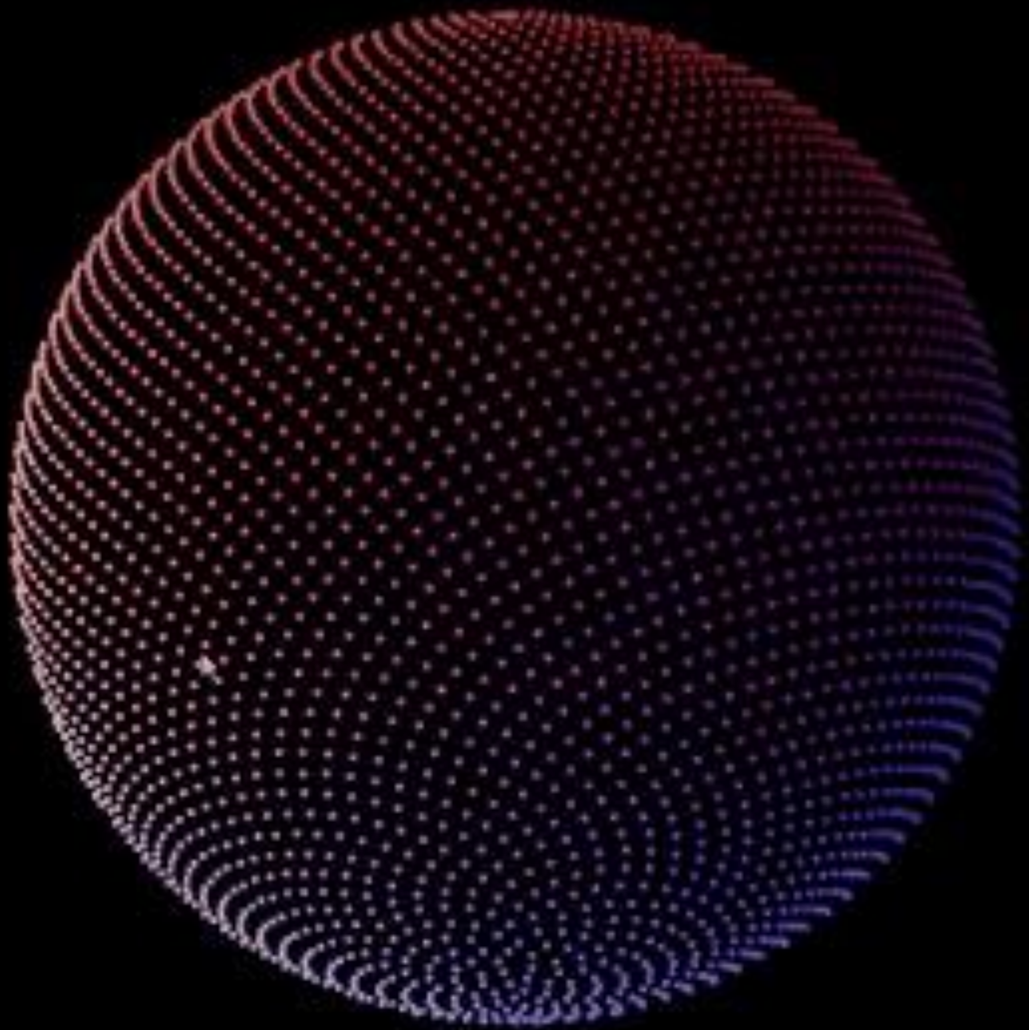


Fig. D





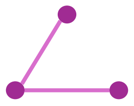
Euler's Formula $v - e + f = 2$

$v = 1$
$e = 0$
$f = 1$

$1 - 0 + 1 = 2$

$v = 2$
$e = 1$
$f = 1$

$2 - 1 + 1 = 2$



$v = 3$
$e = 2$
$f = 1$

$3 - 2 + 1 = 2$

→ Induction

Any questions so far?

Triangulation:
Triangulated Graph / Maximal Planar Graph

Task:

- Determine **which** of these graphs **are triangulated**.
- Develop a formula for the number of edges e, dependent only on the number of vertices v, **within a triangulation.**

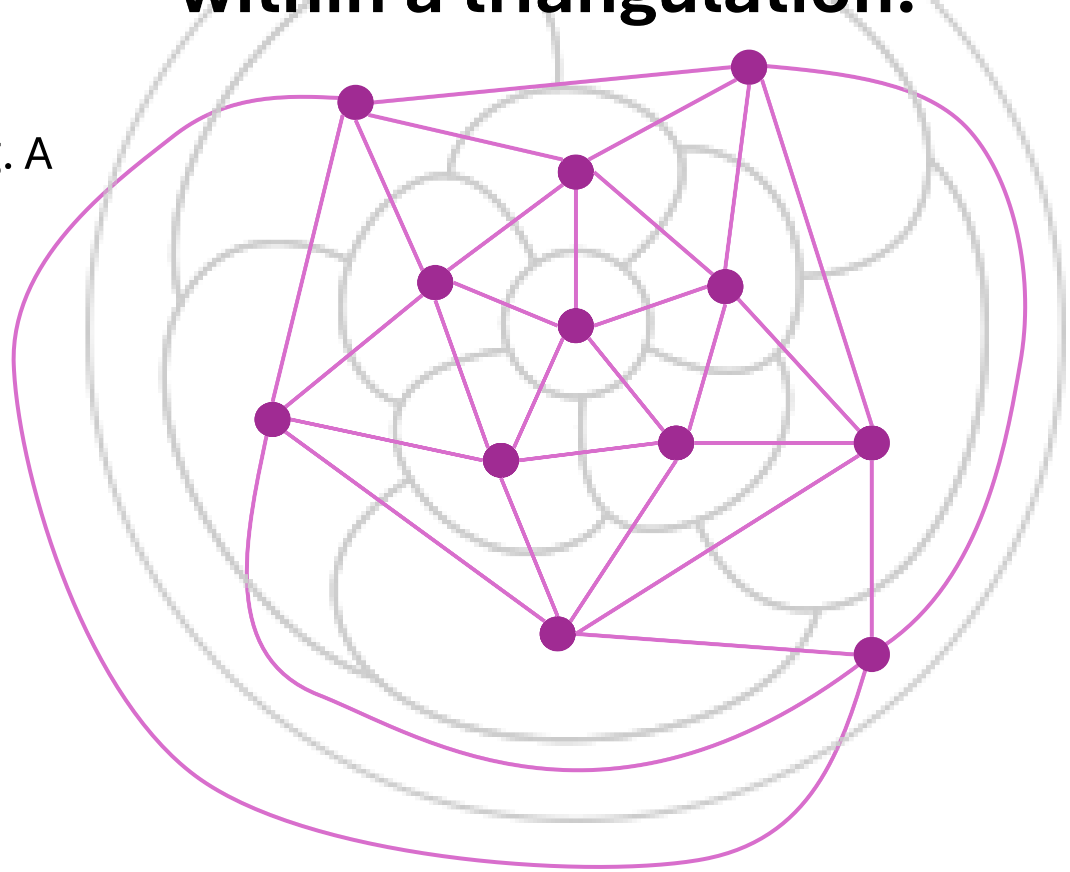


Fig. A

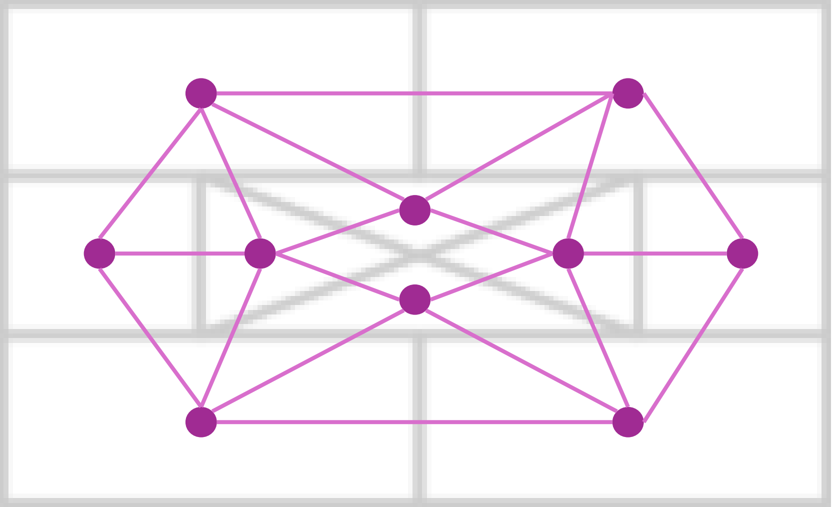


Fig. B

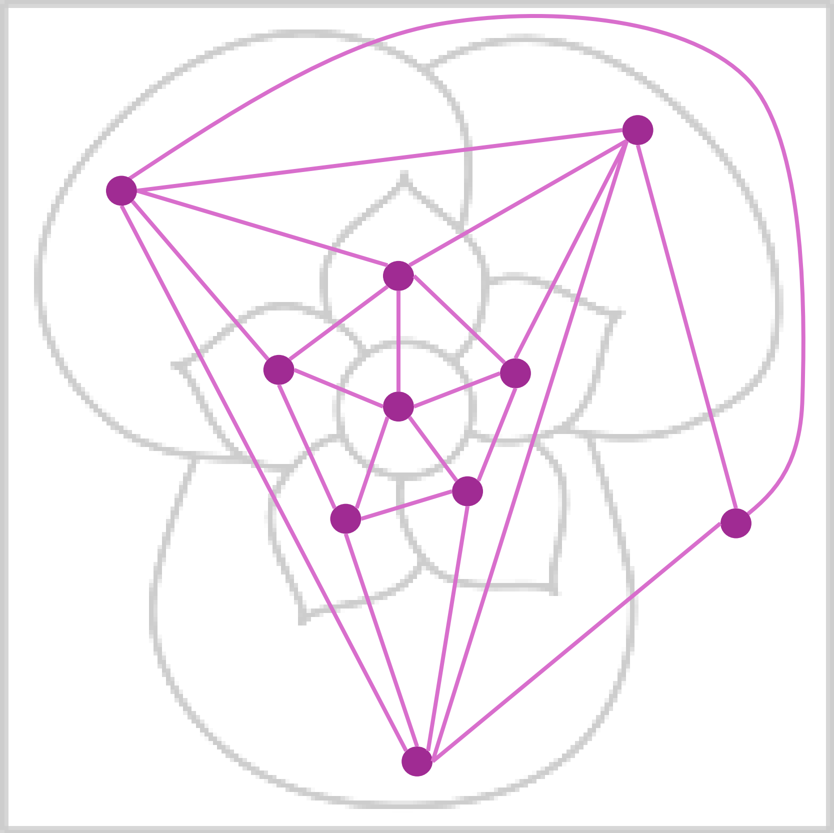


Fig. C

In Triangulation: $e = 3v - 6$

$v = 3$
$e = 3$

$3 = 3 \cdot 3 - 6$

$v = 4$
$e = 6$

$6 = 3 \cdot 4 - 6$

$v = 5$
$e = 9$

$9 = 3 \cdot 5 - 6$

Kempe's Conjecture

In every planar, connected, simple graph, there is at least one vertex with degree five or less.

$$\exists \dot{v} \in V: \deg(\dot{v}) \leq 5$$

$$\exists \dot{v} \in V: \deg(\dot{v}) \leq 5$$

Valid in any triangulation: $e = 3v - 6$

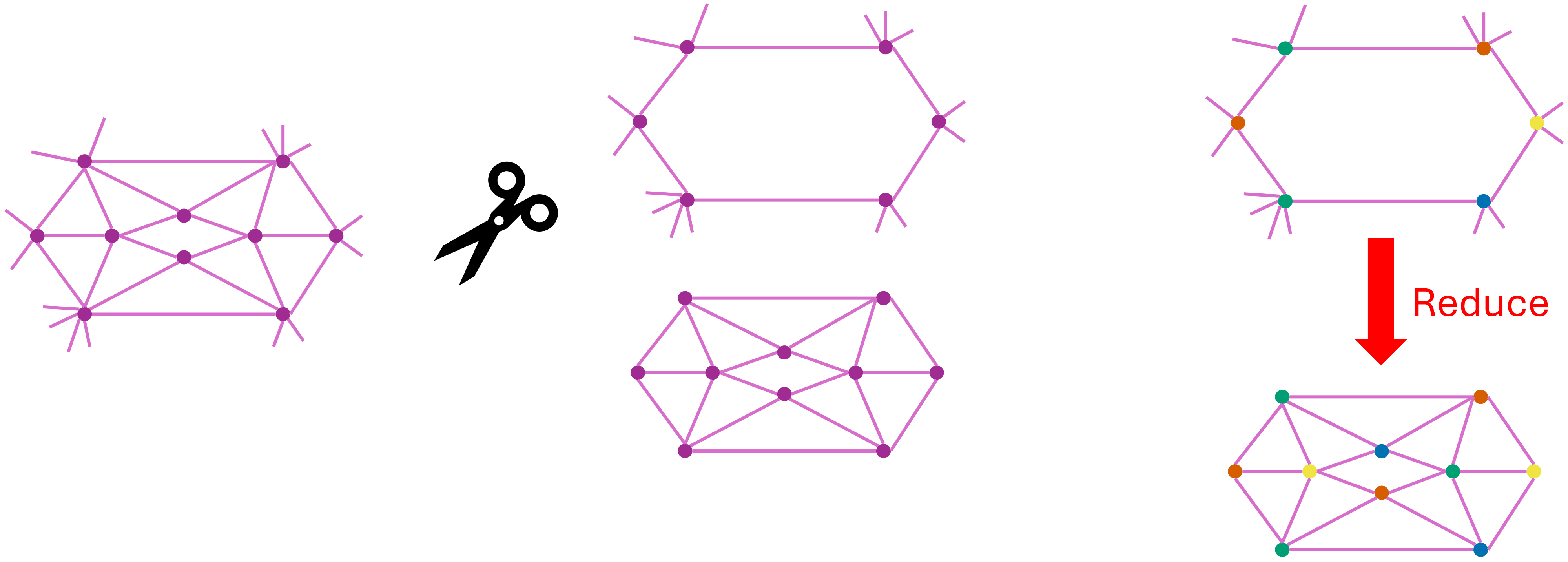
Unavoidable Set

⇒ There is no Graph G that does not contain any of

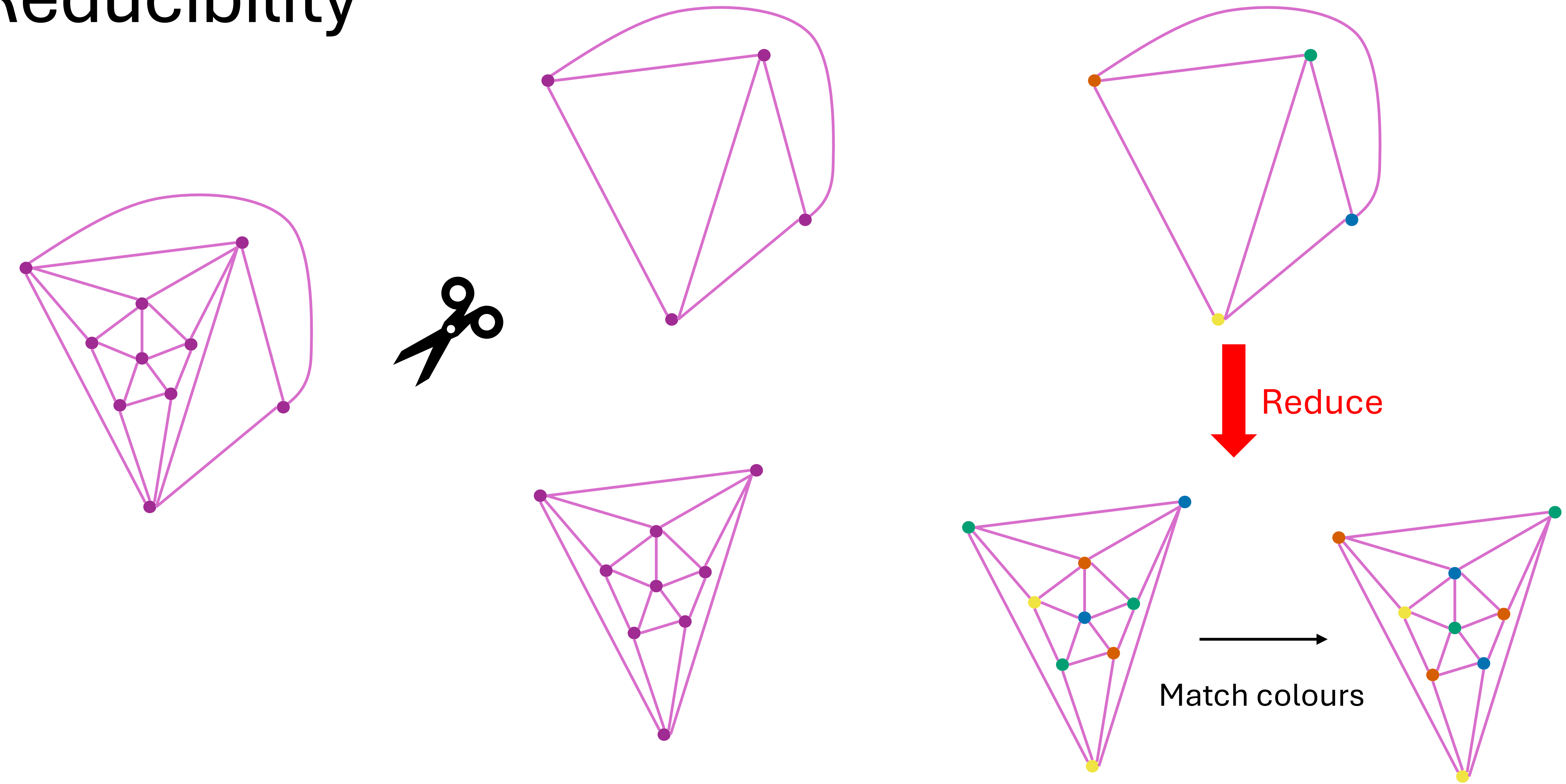
$$U = \{v_1, v_2, v_3, v_4, v_5\}\ with\ \ \deg(v_i) = i, \qquad v_i \in \mathrm{V}$$

⇒ Which of the elements u of U are **REDUCIBLE** ?

Reducibility

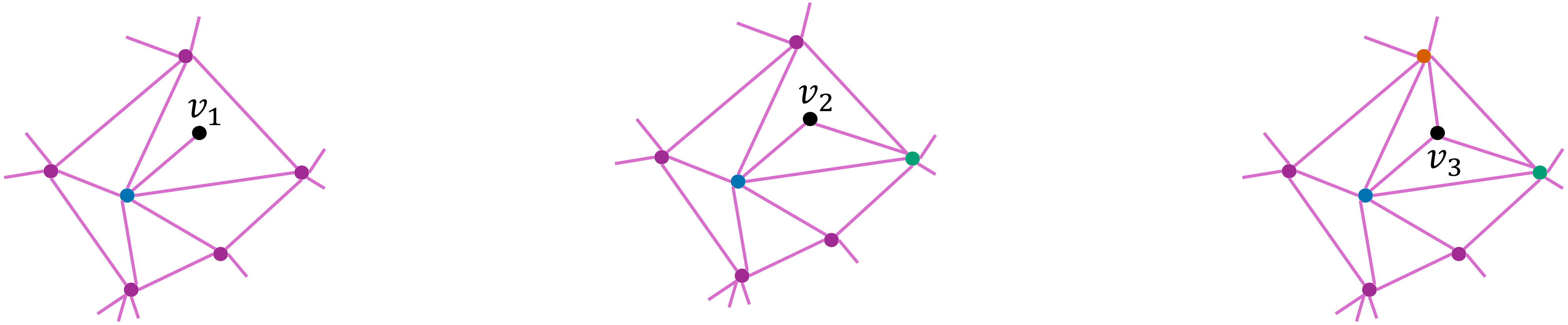


Reducibility



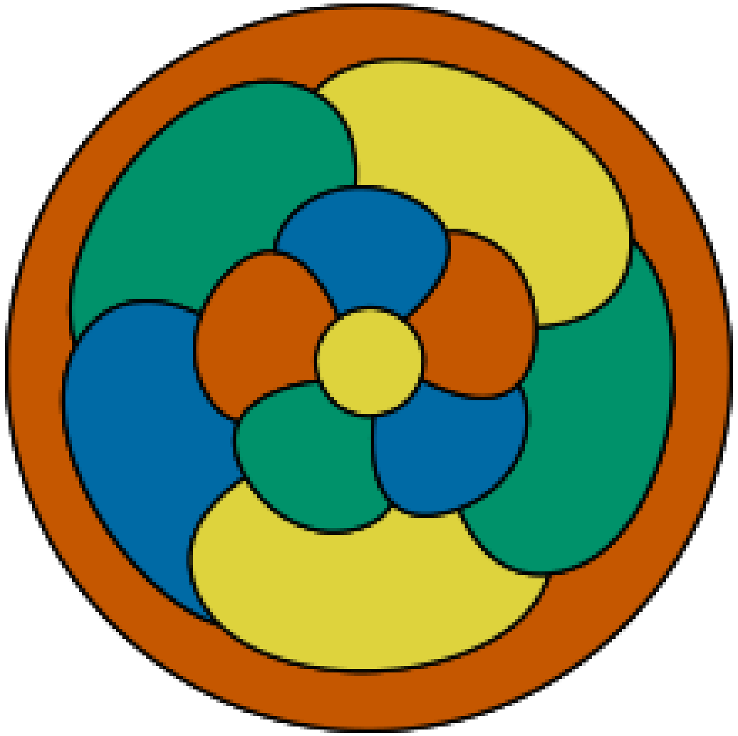
Reduce every element of unavoidable set U

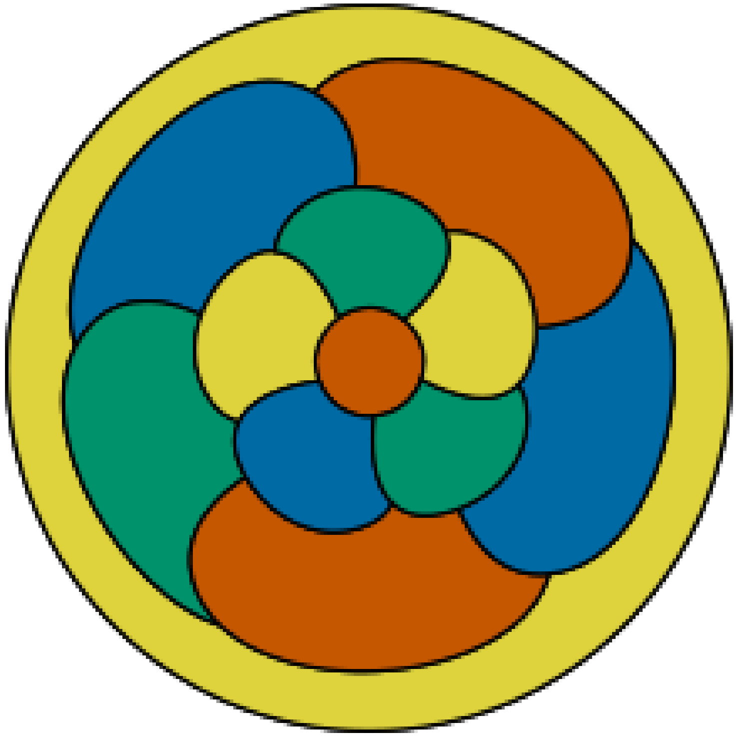
$$U = \{v_1, v_2, v_3, v_4, v_5\}$$



Kempe Chains / Components

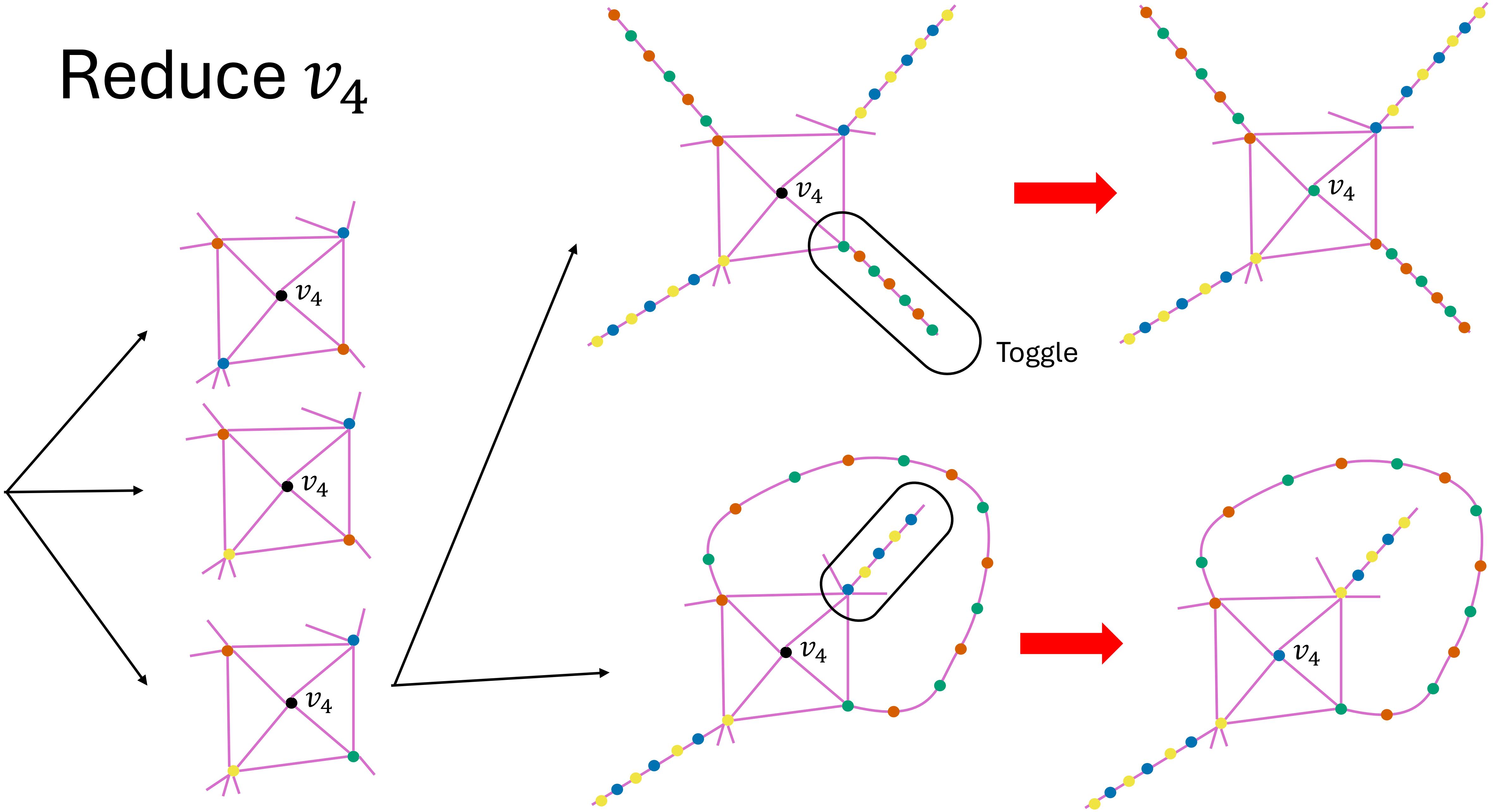
Kempe Chains / Components





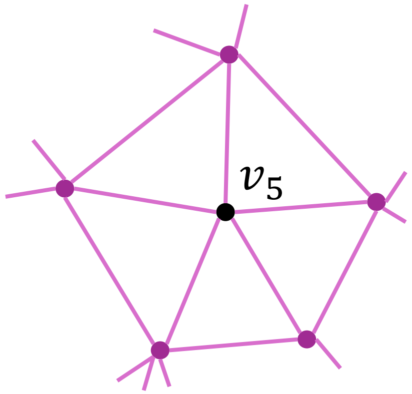
Any questions so far?

Reduce v_4

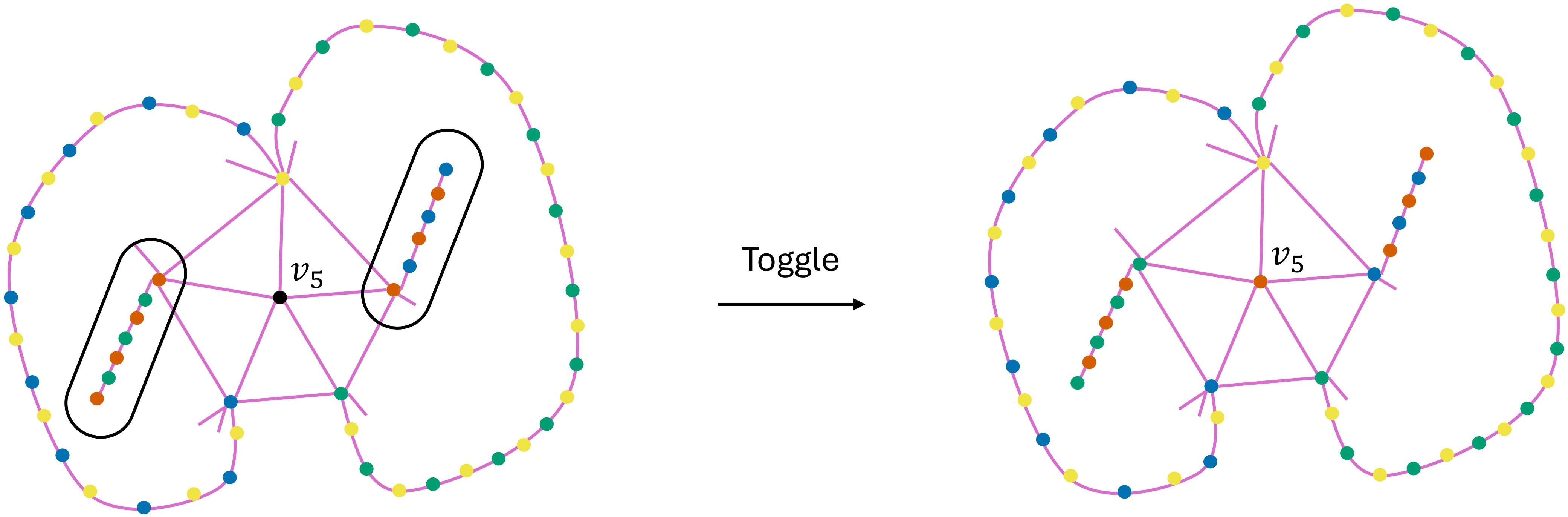


Task:

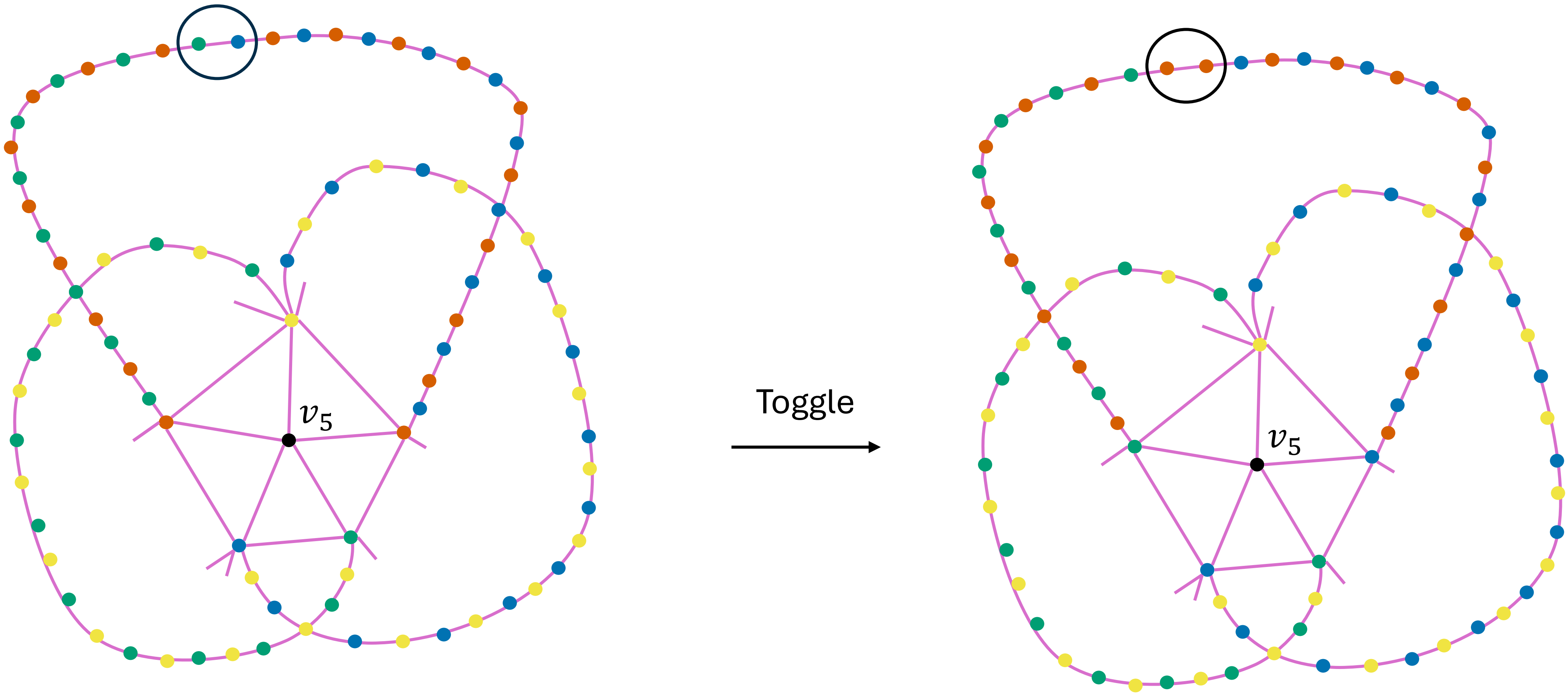
Find a way to reduce v_5 using Kempe-chain-exchanges.

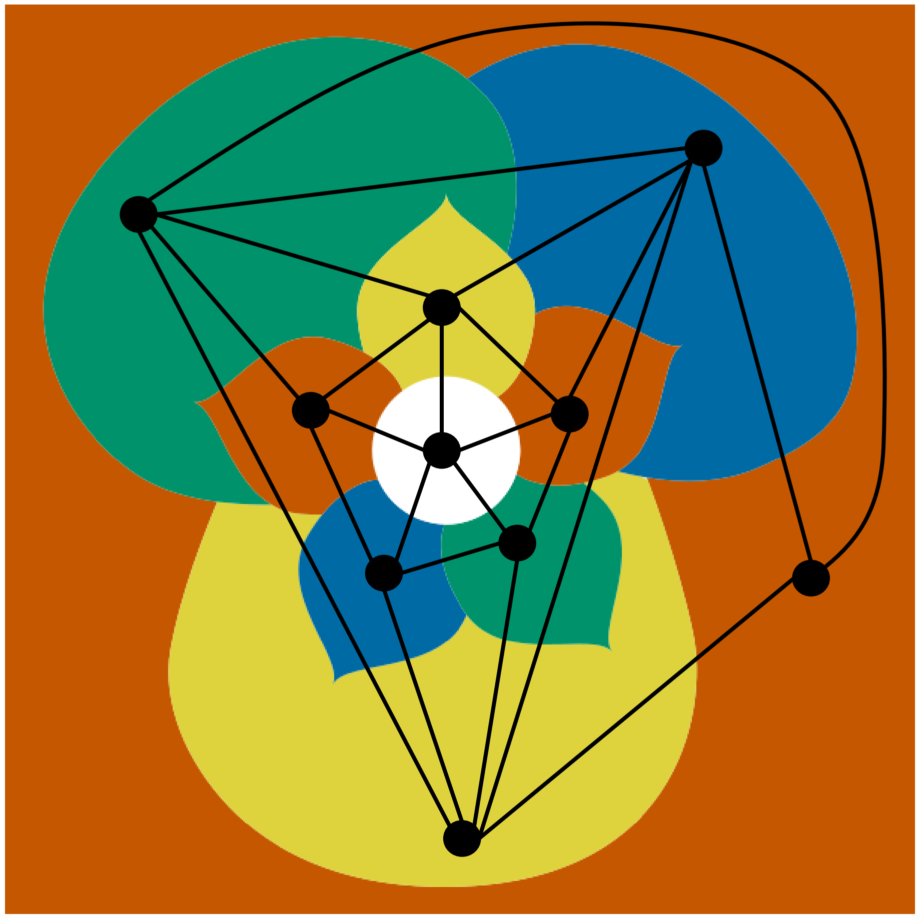


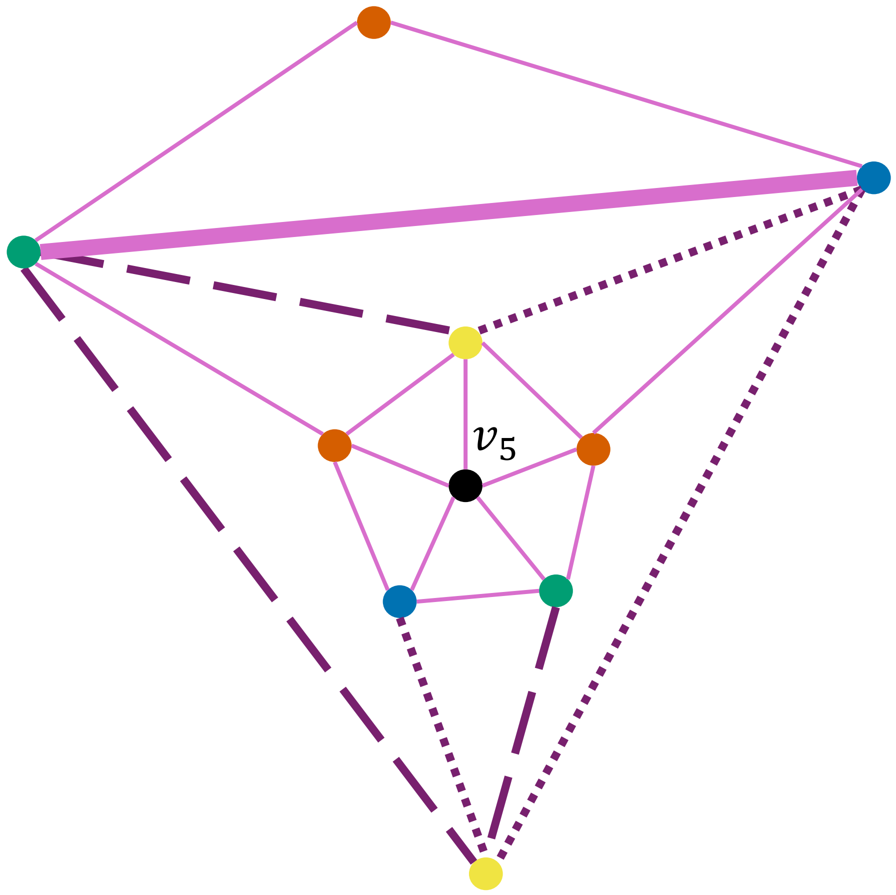
Erroneous proof of v_5-reducibility



Erroneous proof of v_5-reducibility






v_5

$\Rightarrow$ v_5 is not always reducible, so…

New aim:

Find a completely-reducible set U.

"Randomize" a set $U \subset \mathcal{P}(Graphs)$

Reducibility?

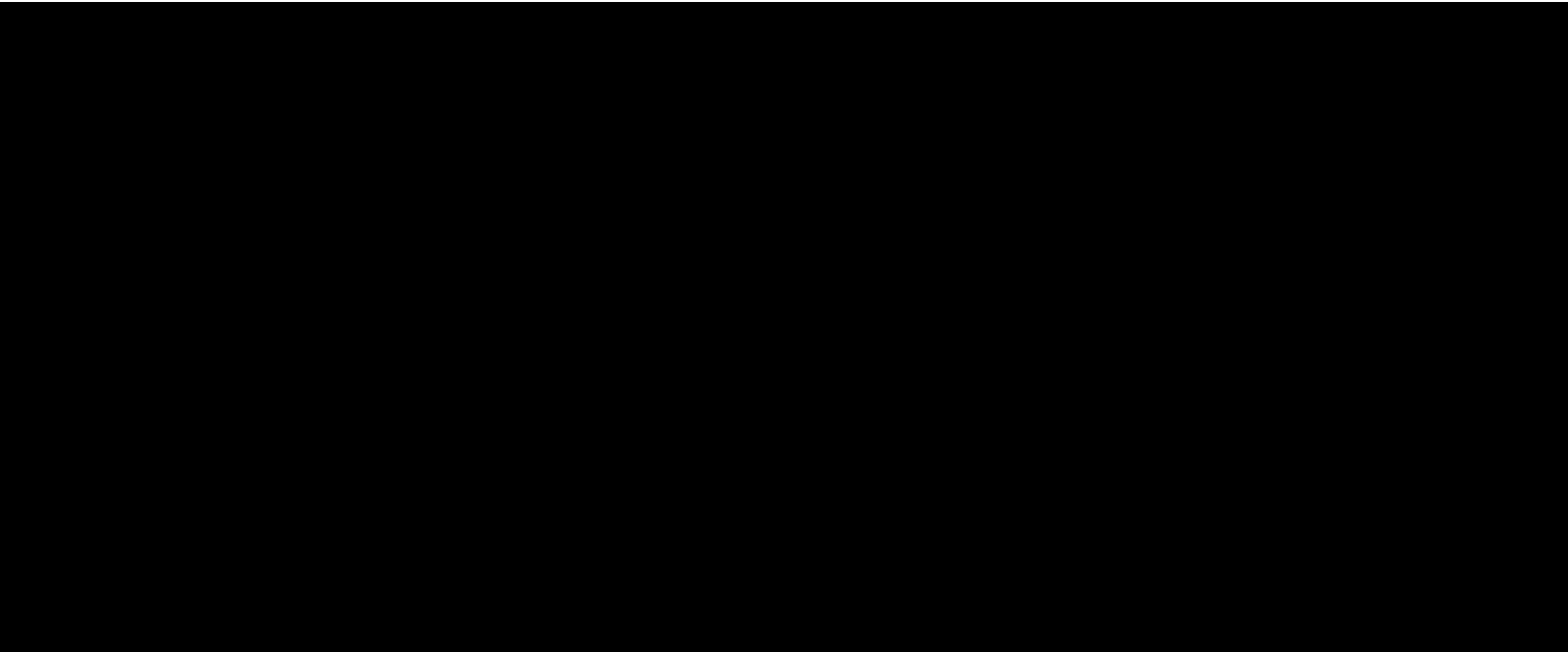
Unavoidability?

Reducibility Algorithms

Discharging Method

Any questions so far?

Unavoidability? → Proof Method: Discharging



```
def check_reducibility_of(graph):
    all_colourings_are_extendible = True
    all_colourings = get_all_possible_colourings_of_outer_ring_of(graph)

    for colouring in all_colourings:
        this_colouring_is_extendible = False
        for pair in [[1,2],[1,3],[1,4]]:
            all_possible_connection_sets = all_possible_connection_sets(graph,pair)
            for connection_set in all_possible_connection_sets:
                new_colouring = modify_colouring_with(graph,colouring,connection_set)
                if extend_colouring_to(graph,new_colouring):
                    this_colouring_is_extendible = True
                    break

        if not this_colouring_is_extendible:
            all_colourings_are_extendible = False
            break
    reducible = all_colourings_are_extendible
    return reducible
```

D-Reducibility

How much “AI” does this proof contain?

References

- **Title picture** Page 1: 'Vier-Farben-Satz'. In *Wikipedia*, 23 October 2024. https://de.wikipedia.org/w/index.php?title=Vier-Farben-Satz&oldid=249668380.
- **Riddle Fig. D** Page 2,3,5,6: 'Four Color Theorem - Coloring Puzzle Game'. Accessed 16 November 2024. https://www.duckaddict.com/four-color-theorem/?v=3.9.
- **Snapshot** Page 8: *Stereographic Projection*, 2013. https://www.youtube.com/watch?v=VX-0Laeczgk.
- **Animation** Page 9: *3D Sphere to 2D Stereographic Projection*, 2020. https://www.youtube.com/watch?v=POUtULNzsQw.
- **All other images, graphs and animations were created by the author.**
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Four Colour Theorem

Any map can be coloured with a maximum of 4 colours in a way that no adjacent regions share the same colour.

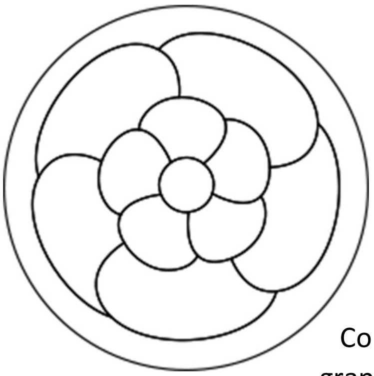
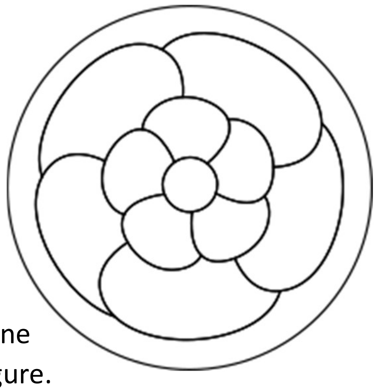


Fig. A



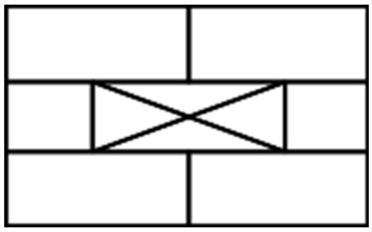


Fig. B

Colorize only one graph of each figure.

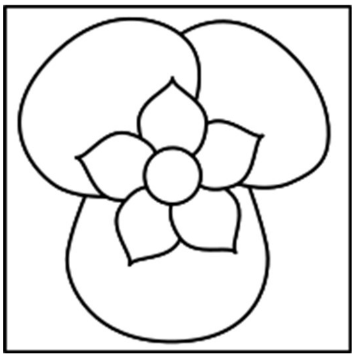
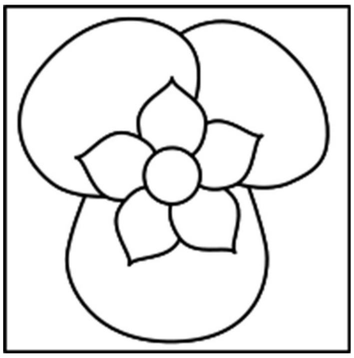


Fig. C



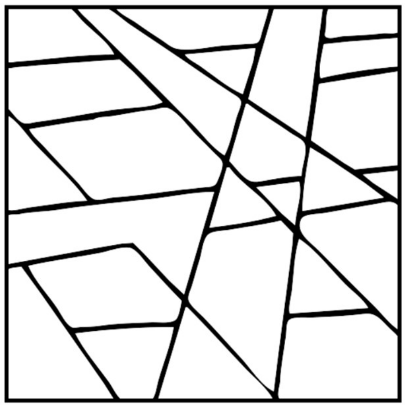


Fig. D – use only 3 colours

Four Colour Theorem

Any map can be coloured with a maximum of 4 colours in a way that no adjacent regions share the same colour.

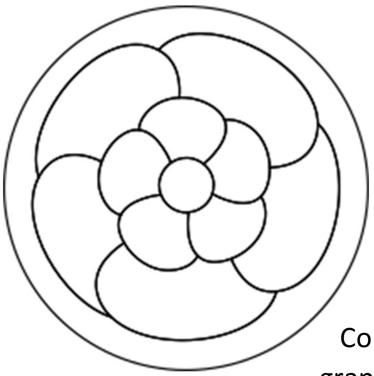
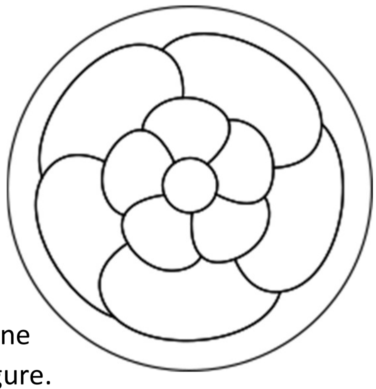


Fig. A



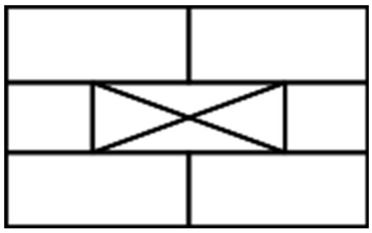


Fig. B

Colorize only one graph of each figure.

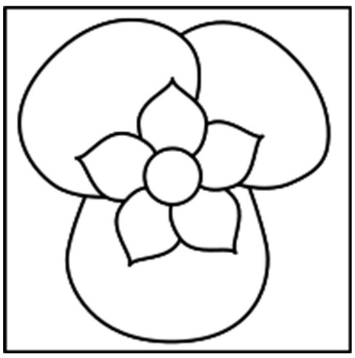
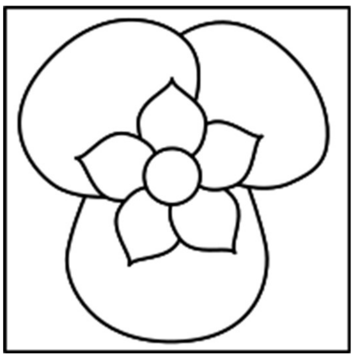


Fig. C



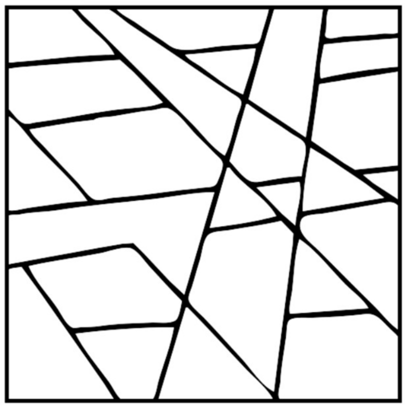


Fig. D – use only 3 colours